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Quantum dynamics and breakdown of classical realism in nonlinear oscillators

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Abstract
The leading nonclassical term in the quantum dynamics of nonlinear oscillators is calculated in the Moyal quasi-trajectory representation. The irreducibility of the quantum dynamics to phase-space trajectories is quantified by the discrepancy of the canonical quasi-flow and the quasi-flow of a general observable. This discrepancy is shown to imply the breakdown of classical realism that can give rise to a dynamical violation of Bell's inequalities.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The behavior of quantum systems in the classical limit is often complicated because many quantum observables become singular in this limit. There is, on the other hand, a large class of observables that have a well-defined classical limit that includes the canonical position, momentum and their powers and products. These observables characterize many important dynamical quantum phenomena and effects, such as the squeezing of uncertainties. By the correspondence principle, the time evolution of the expectation values of the nonsingular observables approaches the classical orbits of the system in the semiclassical limit [1].

The dynamics of the expectation values can of course be expressed either in the Schrödinger picture by evolving the initial state or density matrix in time, or in the Heisenberg picture by evolving the observable of interest. However, the customary formulation of the semiclassical quantum dynamics using time-dependent wavefunctions in the Schrödinger picture [2–5] tends to hide the simple behavior of nonsingular observable expectation values, because wavefunctions are singular in the classical limit. The same problem arises when the dynamics is expressed with the time-dependent Wigner function [6]. Thus, the dynamics of observables is studied here in the Heisenberg picture using the *quasi-flow* [7–9], that is, time-dependent phase-space representations of the observables. This representation avoids semiclassically singular quantities, and leads to an immediate recovery of the classical limit and a straightforward derivation of high order terms in the semiclassical expansion with the help of the Moyal calculus [10].

Here, this method is applied to study the semiclassical dynamics of nonlinear oscillators—dynamical systems defined on the Hilbert space of a single degree of freedom whose classical limit describes oscillations around a single stable fixed point. Such systems not only serve as a paradigm for studying quantum-classical correspondence, but also arise as effective Hamiltonians describing the motion of a particular degree of freedom of interest in a more complex system. One can cite as examples the dynamics of polaritons in strongly coupled cavity QED-atom systems [12] and the relative motion of ions in linear traps [13].

The first purpose of this paper is to use the quasi-flow as a computational tool by deriving concretely and explicitly its semiclassical expansion for a general nonlinear oscillator, with emphasis on the leading quantum term. The immediate application of these results would be a more accurate calculation of dynamical effects like squeezing that are expressed in terms of nonsingular observables. However, the high order terms in the quasi-flow cannot be viewed simply as corrections to the classical flow, because unlike the classical flow, the quasi-flow of a general observable cannot be deduced solely from the canonical quasi-flow. That is, the time evolution classical observable $A(q, p)$, with q and p as the canonical coordinate and momentum, respectively, is given by $A_t(q, p) = A(q_t, p_t)$. This property, which amounts to the fact that classical dynamics is generated by trajectories, does not hold for the quasi-flow (as defined precisely below) when $\hbar$ is nonzero.

Therefore, the high order terms of the quasi-flow derived here quantify the failure of the quantum dynamics to follow a trajectory framework. This failure has observable consequences. Namely, suppose that there is a joint probability distribution for the canonical position and momentum, which reproduces the results of quantum measurements of the initial state in terms of a classical hidden variable [14, 15]. In other words, classical realism can be assigned initially to the oscillator. Such a hidden variable underpinning can be supplied by the initial state Wigner function if it is non-negative, like the Wigner function of a coherent state. Then [16], under the nonlinear time evolution the hidden variable underpinning will cease to be valid, and this failure can be observed by the violation of Bell inequalities if the oscillator is initially entangled with another system. This effect, which can be described as a dynamical breakdown of the initial classical realism, is studied here in the semiclassical limit, and its quantification using the high order terms in the quasi-flow is the second goal of this paper.

The results of this paper are as follows: After reviewing the properties of the Moyal quasi-flow, its semiclassical expansion for a broad class of nonlinear oscillators is derived, and the explicit expressions for the leading quantum term are shown in equations (4)–(6). Apparently, this is the first time that nonclassical terms in the quasi-trajectories of a nonlinear system have been calculated explicitly for long evolution times—up to the Heisenberg time. As explained above, a key distinguishing property of the high order terms in the quasi-flow is that they cannot be deduced from the quasi-flow of the canonical observables; therefore, the canonical quasi-flow has to be supplemented by information about the discrepancy of the quasi-flow of a general operator with respect to the flow which would be generated for this operator by the canonical quasi-flow. The leading term in the semiclassical expansion of this discrepancy is derived in the form of a generating function presented in equation (9). These results are next applied to study the dynamical breakdown of classical realism in the semiclassical limit, by calculating the quasi-flow of a bounded observable. In the framework of a hidden variable underpinning implied by non-negative Wigner function initial states, it is shown that the conditions necessary for the persistence of the hidden variable underpinning are not met (equation (10)) and how this failure may lead to dynamical Bell inequality violations. The concepts and results are derived for an oscillator with an arbitrary form of nonlinearity;

as an example, they are applied to the Hamiltonian of a single electromagnetic mode with a Kerr nonlinearity.

The present study of nonlinear oscillator dynamics is greatly simplified by the recently developed normal form theory of nonlinear oscillators [17, 18], according to which every nonlinear oscillator can be unitarily transformed into a normal form which commutes with the harmonic oscillator Hamiltonian. The dynamical analysis is carried out for a nonlinear oscillator in a normal form. The calculational details required to apply the results to nonlinear oscillators in a general form are presented in the appendix.

2. The Moyal quasi-flow

As is well known, although position and momentum are not simultaneously measurable in quantum mechanics, it is still possible to define phase-space quasi-probabilites [19]. The quasi-probabilities have to be accompanied by a symbol map, which assigns a phase-space function to each Hilbert space operator, such that the expectation value of the operator is equal to the phase-space integral of the symbol with the quasi-probability as a measure.

A natural alternative to the representation of dynamics by time-dependent wave or Wigner functions is to use quasi-trajectories [7] defined by the symbols $(q_t(z), p_t(z))$ of the canonical position $\hat{q}_t$ and momentum $\hat{p}_t$ in the Heisenberg picture (we use the symplectic notation $z^1 = -z_2 = q, z^2 = z_1 = p$ with implied summation over repeated indices); at a given time they define a mapping $z_t(z)$ from the classical phase plane to itself which will be termed the quasi-flow. Since the symbols of $\hat{q}$ and $\hat{p}$ are the functions q and p (respectively), the initial value of the quasi-flow is the identity mapping. The quasi-trajectories facilitate the calculation of expectation values of the canonical variables as a function of time for any initial state; the quasi-flow $A_t(q, p)$ of a general observable is defined similarly as the phase-space representation of the observable $\hat{A}_t$ in the Heisenberg picture.

The distinct advantage of using quasi-trajectories over more common semiclassical methods is the harmless nature of the $\hbar \to 0$ limit, where the quasi-trajectories simply reduce to the classical trajectories. Moreover, the calculation of higher order terms in the semiclassical expansion for the quasi-trajectories and the phase-space representation of other Heisenberg-picture observables is straightforward. The same simplifying property holds for thermal state Wigner functions [20], but not for pure states [21].

Evidently, the form of the quasi-flow depends on the choice of the quasi-probability and symbol scheme. Here we choose the Wigner–Weyl representation [20, 22], since its convenient properties make it particularly suitable for the analysis of the quasi-flow.

3. The canonial quasi-flow

Nonlinear oscillators are defined here as (spinless) one-degree of freedom quantum dynamical systems governed by Hamiltonians whose Weyl symbol, the classical Hamiltonian $H(q, p)$, has a single nondegenerate phase-space minimum and no other fixed points. It has been demonstrated in [17] that such Hamiltonians are unitarily equivalent to a normal form Hamiltonian which commutes with the harmonic oscillator Hamiltonian. That is, there exists a unitary operator $\hat{U}$ such that

$$\hat{Q} = \hat{U}^\dagger \hat{q} \hat{U}, \qquad \hat{P} = \hat{U}^\dagger \hat{p} \hat{U} \tag{1}$$

and

$$\hat{H} = f(\hat{B}), \qquad \hat{B} = \frac{\hat{Q}^2 + \hat{P}^2}{2}. \tag{2}$$

The classical limit of this transformation is a Cartesian version of the canonical transformation to action-angle variables; in particular the classical limit of B, the symbol of $\hat{B}$, is the classical action. Like its classical analog, the quantum normal form Hamiltonian is a representative of its equivalence class where dynamics takes a particularly simple form. For this reason we will assume here that the Hamiltonian has been reduced to a normal form, construct its quasi-flow and study its properties. The application of the results to nonlinear oscillators in a general form is dealt with in the appendix.

It is convenient to formulate the normal form dynamics in terms of the complex coordinate $\beta = \frac{1}{\sqrt{2}}(Q + \mathrm{i}P)$ and the annihilation operator $b = \frac{1}{\sqrt{2}}(\hat{Q} + \mathrm{i}\hat{P})$ (note that b differs by a factor of $\sqrt{\hbar}$ from the usual definition). The symbol of b_t will be denoted by β_t. In complex coordinates, $B = |\beta|^2$ and $\hat{B} = \frac{1}{2}(bb^\dagger + b^\dagger b)$.

Using the algebraic properties of the observables [23], their dynamics in a normal form can be expressed as a rotation with an operator-valued frequency

$$b_t \equiv \mathrm{e}^{-\frac{\hat{H}}{\mathrm{i}\hbar}t} b\, \mathrm{e}^{\frac{\hat{H}}{\mathrm{i}\hbar}t} = \mathrm{e}^{-\mathrm{i}\Omega_+(\hat{B})t} b = b\, \mathrm{e}^{-\mathrm{i}\Omega_-(\hat{B})t}, \tag{3}$$

where $\Omega_\pm(\hat{B}) = \pm\frac{1}{\hbar}(f(\hat{B} \pm \hbar) - f(\hat{B}))$. Note that $\lim_{\hbar\to 0}\Omega_\pm = \omega$ is the classical (action-dependent) frequency.

Either of the two expressions on the right-hand side of equation (3) forms a useful starting point for a semiclassical expansion, since their factors have symbols with smooth dependence on $\hbar$. This fact makes the calculation of the semiclassical expansion of the normal-form quasi-flow a straightforward application of the rules of the Moyal calculus; the resulting expression to the second order in $\hbar$ is

$$\beta_t(\beta, \bar{\beta}) = \beta_t^{(c)}\big(1 + \hbar^2 \beta_t^{(2)} + O(\hbar^4)\big), \tag{4}$$

where $\beta_t^{(c)} = \beta\, \mathrm{e}^{-\mathrm{i}\omega(B)t}$ is the classical flow, and

$$\mathrm{Re}\, \beta_t^{(2)} = \tfrac{1}{4}((\omega'(B)t)^2 + B\omega'(B)\omega''(B)t^2), \tag{5}$$

$$\mathrm{Im}\, \beta_t^{(2)} = \tfrac{1}{24}(-2B(\omega'(B)t)^3 + 5\omega''(B)t + 2B\omega'''(B)t) \tag{6}$$

are the leading quantum corrections to the modulus and phase of β_t, respectively. Snapshots of the coordinate quasi-flow and its deviation from the classical flow are shown in figure 1 for a Kerr Hamiltonian.

By the phase-space correspondence principle, the expectation values of observables that have a well-behaved classical limit are given by the phase-space integral of the corresponding time-dependent classical observable with the initial Wigner function as a measure. Equation (4) establishes the correspondence principle for the position and momentum, and equations (5) and (6) express the drift generated by the quantum nonlinearities. More importantly, the higher order terms in the quasi-flow capture the fundamental difference between classical and quantum nonlinear dynamics. The implications of this fact are explored in the following sections.

A remark is in order regarding the amplitude drift implied by the nonvanishing of $\mathrm{Re}\, \beta_t^{(2)}$, since energy conservation implies conservation of the action operator $\hat{B}$. Rather than implying that $\mathrm{Re}\, \beta_t^{(2)} = 0$, conservation of energy determines $\mathrm{Re}\, \beta_t^{(2)}$ solely on the basis of the classical flow. Thus, on one hand $B_t(\beta) = B$, and on the other hand

$$B_t(\beta) = \frac{\beta_t \star \bar{\beta}_t + \bar{\beta}_t \star \beta_t}{2} = B + \hbar^2 \left(2B\beta_t^{(2)} - \frac{1}{8}\partial_\mu \partial_\nu \beta_t^{(c)} \partial^\mu \partial^\nu \bar{\beta}_t^{(c)}\right) + O(\hbar^4), \tag{7}$$

where $\star$ stand for the Moyal product, $\partial_\mu = \frac{\partial}{\partial z^\mu}$, and $\partial^\mu = -\frac{\partial}{\partial z_\mu}$. The equality of the two expressions for B_t implies that $\mathrm{Re}\, \beta_t^{(2)} = \frac{1}{16B}\partial^\mu \partial^\nu \beta_t^{(c)} \partial_\mu \partial_\nu \bar{\beta}_t^{(c)}$, consistently with equation (5).

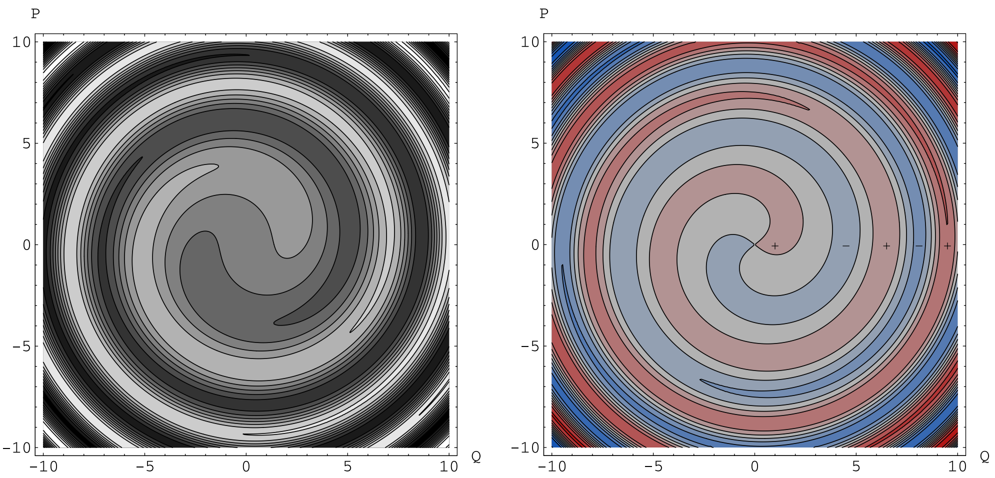

Figure 1. Left: contour plot of a snapshot of the second-order semiclassical approximation to the coordinate quasi-flow $Q_t(Q, P)$ for a nonlinear oscillator with a Hamiltonian $H = \omega_0 \hat{B} + \frac{1}{2}\omega_1 \hat{B}^2$, $\frac{\omega_1 \hbar}{\omega_0} = 0.04$. The snapshot is taken at $t = 2\pi/\omega_0$, and the Q, P coordinates are shown in units of $\sqrt{\hbar}$. Larger values are associated with lighter shades of grey. Right: the leading quantum correction to the coordinate quasi-flow $Q_t^{(2)}(Q, P)$ divided by $\sqrt{B}$, for the same system and time. Positive value regions are marked with '+' signs and shown in red hues and negative regions are marked with '−' and marked in blue; the color intensity encodes the absolute value, with maximal values near 20%.

A more fundamental issue regarding the semiclassical expansion of the quasi-flow is its nonuniformity in time. It is evident from the explicit expressions (4)–(6) that the expansion, which is derived for a fixed time interval, ceases to be valid at late enough times. In other words, the limits $\hbar \to 0$ and $t \to \infty$ do not commute, a manifestation of a general property [21]. In particular, the expansion based on classical dynamics cannot capture wavefunction revival. Nevertheless, the expansion remains valid for times which are larger than any classical time scale; see figure 1 for an example. It follows from equations (4)–(6) that the time scale of validity is the Heisenberg time $(\hbar\omega')^{-1}$; the typical number of oscillations in the interval of validity is $B_0/\hbar$, where $B_0 = \omega/\omega'$ is the typical action scale of nonlinearity.

4. The quasi-flow of a general observable

Classical dynamics is fully described by a set of trajectories $z_t^{(c)}(z)$ that gives the time evolution of classical phase-space distributions or, alternatively, of classical observables. This is no longer true for nonlinear quantum dynamics, a manifestation of the impossibility of assigning a definite instantaneous position and momentum values to a quantum particle.

In the semiclassical limit the trajectory framework still holds approximately, though. That is, if an observable $\hat{A}$ is represented by the symbol $A(\beta, \bar{\beta})$, and its time evolution $\hat{A}_t$ by A_t, then

$$d_t(\hat{A}) = A_t(\beta) - A(\beta_t) = O(\hbar^2). \tag{8}$$

$d_t(\hat{A})$ will be termed the discrepancy of the observable $\hat{A}$ at time t, since it measures the failure of the quantum dynamics to follow trajectories. As such, it will be used in section 5 as a diagnostic for the departure from classical realism.

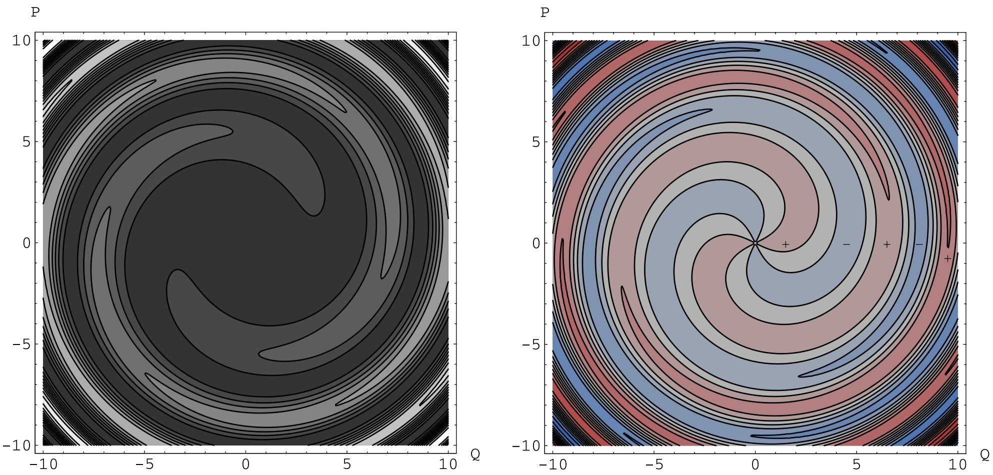

Figure 2. Left: contour plot of a snapshot of the second-order semiclassical approximation to the quasi-flow of the square of the coordinate $Q_t^2(Q, P)$ for the nonlinear oscillator used in figure 1 at time π/ω_0. Larger values are associated with lighter shades of grey. Right: the leading term discrepancy $d_t(Q_t^2)$ divided by B, for the same system and time. The coding system is the one used for the right panel of figure 1, and the maximal values are near 7%.

Table 1. Coefficients in the leading term of the discrepancy of quasi-flows associated with the generating function, equation (9).

$$P_1 = \tfrac{1}{4}\mathrm{i}\bar{\alpha}\bar{\beta}_t^{(c)}\alpha^2 + \tfrac{1}{4}\mathrm{i}\bar{\alpha}^2\beta_t^{(c)}\alpha$$

$$P_2 = -\tfrac{1}{8}\bar{\alpha}^3(\beta_t^{(c)})^3 + \tfrac{1}{2}\bar{\alpha}^2(\beta_t^{(c)})^2 + \tfrac{1}{8}\alpha^3(\bar{\beta}_t^{(c)})^3 + \tfrac{1}{2}\alpha^2(\bar{\beta}_t^{(c)})^2 + \tfrac{1}{2}B\alpha\bar{\alpha} - \tfrac{1}{8}B\alpha\bar{\alpha}^2\beta_t^{(c)} + \tfrac{1}{8}B\alpha^2\bar{\alpha}\bar{\beta}_t^{(c)}$$

$$P_3 = \tfrac{1}{12}\mathrm{i}B\bar{\alpha}^3(\beta_t^{(c)})^3 + \tfrac{1}{12}\mathrm{i}B\alpha^3(\bar{\beta}_t^{(c)})^3 - \tfrac{1}{4}\mathrm{i}B\bar{\alpha}^2(\beta_t^{(c)})^2 + \tfrac{1}{4}\mathrm{i}B\alpha^2(\bar{\beta}_t^{(c)})^2 + \tfrac{1}{4}\mathrm{i}B^2\alpha\bar{\alpha}^2\beta_t^{(c)} + \tfrac{1}{4}\mathrm{i}B^2\alpha^2\bar{\alpha}\bar{\beta}_t^{(c)}$$

$$P_4 = \tfrac{1}{24}\mathrm{i}\bar{\alpha}^3(\beta_t^{(c)})^3 + \tfrac{1}{8}\mathrm{i}B\alpha\bar{\alpha}^2\beta_t^{(c)} + \tfrac{1}{24}\mathrm{i}\alpha^3(\bar{\beta}_t^{(c)})^3 + \tfrac{1}{8}\mathrm{i}B\alpha^2\bar{\alpha}\bar{\beta}_t^{(c)}$$

$$P_5 = \tfrac{1}{4}B\bar{\alpha}^2(\beta_t^{(c)})^2 + \tfrac{1}{4}B\alpha^2(\bar{\beta}_t^{(c)})^2 + \tfrac{1}{2}B^2\alpha\bar{\alpha}$$

The discrepancy of a general observable in the $\hat{Q}$, $\hat{P}$ algebra is extractable from the discrepancy of the generating function $\mathrm{e}^{\alpha b_t^\dagger - \bar{\alpha} b_t}$ of the complex parameter α, which is calculated by another application of the Moyal calculus,

$$d_t\left(\mathrm{e}^{\alpha b_t^\dagger - \bar{\alpha} b_t}\right) = \hbar^2 \mathrm{e}^{\alpha\bar{\beta}_t^{(c)} - \bar{\alpha}\beta_t^{(c)}}(\omega'(B)t P_1 + (\omega'(B)t)^2 P_2 + (\omega'(B)t)^3 P_3 + \omega''(B)t P_4 + \omega'(B)\omega''(B)t^2 P_5) + O(\hbar^4), \tag{9}$$

where $P_1, \ldots, P_5$ are low-degree polynomials in B, $|\alpha|^2$, $\alpha\bar{\beta}_t^{(c)}$ and $\bar{\alpha}\beta_t^{(c)}$, whose explicit form is shown in table 1. Together with the quasi-flow, equation (9) gives full information on the quantum dynamics of the (normal-form) nonlinear oscillator to $O(\hbar^4)$.

A snapshot of the quasi-flow for the square of the coordinate and its discrepancy is shown in figure 2 for the same Hamiltonian used in figure 1. The example shows the result of evolution for a period where classical nonlinearities are strong, but the quantum effects, such as the discrepancy are still small.

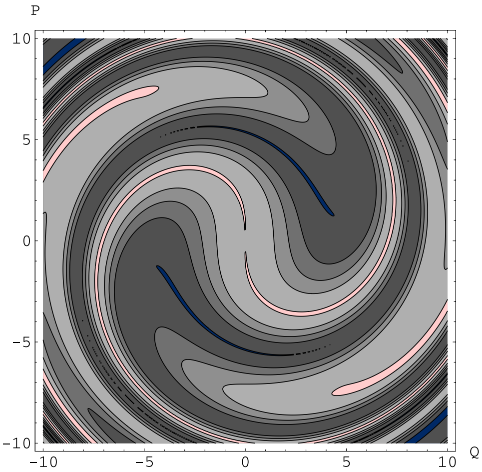

Figure 3. Contour plot of a snapshot of the second-order semiclassical approximation to the quasi-flow of the observable $C = \cos(\alpha\hat{P}), \alpha = \pi/(6\sqrt{\hbar})$, for the nonlinear oscillator used in figure 1 at time $\pi/(2\omega_0)$. The colored portions show regions of breakdown of realism by the Weyl–Wigner representation, where $C_t > 1$ (light red) or $C_t < -1$ (dark blue).

5. Breakdown of classical realism by quantum nonlinear dynamics

The assignment of classical realism (or a hidden variable underpinning) to a quantum state consists of a probability density $\rho(\lambda)$, and representations $q(\lambda)$ and $p(\lambda)$, which describe the statistics of the position and momentum (respectively) using the hidden variable λ. The Wigner function of a state serves as a hidden variable underpinning if it is non-negative. It also satisfies Bell's locality postulates, and was therefore used by Bell to show that his inequalities hold for an Einstein–Podolosky–Rosen-like entangled state [14]. It has been shown later [24] that Bell's argument contained an implicit assumption, and that it is possible to construct Bell operators which violate the inequalities even when the Wigner function is positive. As explained in [16], the additional assumption needed to derive Bell's inequalities relates to the bounded local observable from which the Bell operator is constructed. Namely the hidden variable representation $A(\lambda)$ of such an observable $\hat{A}$ should be bounded by $\|\hat{A}\|$, the norm of $\hat{A}$. Observables which satisfy this condition will be termed *proper*.

When a system-measurement setup which initially supports a local hidden variable underpinning is subjected to nonlinear evolution, the failure of the quantum dynamics to follow trajectories implies that the classical realism inherent in the initial setup breaks down, a fact which may be detected by the violation of Bell's inequalities. When the hidden variable underpinning is supplied by a non-negative Wigner function, the breakdown of classical realism is manifest either by the appearance of negative values for the Wigner function (in the Schrödinger picture) or by the appearance of values of the Weyl representation of the local bounded observables not bounded by the norm of the observable (in the Heisenberg picture) [16].

The dynamical analysis presented above is naturally applied to study this effect in the semiclassical limit. Specifically, the discrepancy of the quasi-flow of an initially proper observable with respect to the canonical quasi-flow, defined in section 4, is directly related to its failure to be proper at later times.

As an example for a bounded observable consider the observable $\hat{C} = \cos\alpha\hat{P}$, which is bounded by 1; its symbol is $C = \cos(\alpha P)$ (in real coordinates) so that $\hat{C}$ is a proper observable. By the previous arguments we expect that $\hat{C}_t$ becomes improper for positive time, i.e. that there exist phase-space points where its symbol no longer satisfies $|C_t| \leqslant 1$. The regions where the realism inequality $|C_t| \leqslant 1$ is liable to fail are where it is already saturated classically, for example at points mapped by the classical flow to the $P = 0$ line. Indeed, by equation (9)

$$C_t\big(\beta^{(c)}_{-t}(Q,0)\big) = 1 + \hbar^2\frac{\alpha^2}{8}\left(3\omega'\left(\frac{Q^2}{2}\right) + Q^2\omega''\left(\frac{Q^2}{2}\right)\right)\omega'\left(\frac{Q^2}{2}\right)t^2 + O(\hbar^4), \tag{10}$$

where $\beta^{(c)}_{-t}(Q,0)$ is the point mapped by the classical flow to $(Q,0)$ at time t. For positive times this expression is larger than 1 for a large class of nonlinear oscillators, so that $\hat{C}_t$ becomes improper under these nonlinear dynamics; the regions of failure of classical realism for $\hat{C}_t$ under a Kerr nonlinearity are shown in figure 3.

As explained above, this result implies the possibility of Bell inequality violations in an entangled state with a non-negative Wigner function. Bell measurements constructed from the complementary observables $\cos(\alpha\hat{q})$ and $\cos(\alpha\hat{P})$ (for each party) are guaranteed to satisfy Bell's inequalities initially. However, if one or both of the parties undergoes dynamics in a nonlinear oscillator prior to the measurement of the Bell operators, then Bell's inequalities are liable to be violated according to equation (10), and the amount of violation increases with time. Further study of this topic is beyond the scope of the present work.

6. Conclusions

The semiclassical expansion of the Moyal quasi-trajectories is a nonsingular series in powers of $\hbar$. As such, it does not easily capture quantum features of the dynamics which are singular in the classical limit, like the evolution of interference patterns; on the other hand it provides a direct route to study dynamical quantities which depend smoothly on $\hbar$, without the need to go through painstaking stationary phase calculations just to recover the leading classical term, and thereby explore the consequences of quantum nonlinear evolution in the semiclassical limit.

The straightforward expansion derived here for nonlinear oscillators holds for a finite time interval, and is nonuniform in time, a universal property of dynamical semiclassical approximations, which stems from the different asymptotic nature of the classical and quantum dynamics. Nonetheless, preliminary results indicate that the semiclassical approximation to the quasi-flow can be extended beyond the Heisenberg time; in this case, however, the classical limit would be once more singular.

Acknowledgments

I have benefited from fruitful discussions with Y Avron, M Berry, R Littlejohn and R Zeitak.

Appendix. Dynamics of nonlinear oscillators in a general form

Since every nonlinear oscillator is unitarily equivalent to a normal form oscillator [17], the results presented above yield the quasi-flow of any given nonlinear oscillator after it has been transformed to its normal form. Still, it is also of interest to express the result in terms of the original observables $\hat{q}$, $\hat{p}$ rather than the transformed observables $\hat{Q}$, $\hat{P}$.

In the classical limit, the nonlinear oscillator flow can be expressed as a composition of three canonical transformations $(q, p) \longrightarrow (A, \phi) \longrightarrow (A, \phi + \omega(A)t) \longrightarrow (q_t, p_t)$, where the first and last arrows are the transformation to action-angle variables and its inverse (respectively), and the middle arrow is the time-evolution canonical transformation. Analogously the quantum evolution is the composition of three unitary transformations, $(\hat{q}, \hat{p}) \longrightarrow (\hat{Q}, \hat{P}) \longrightarrow (\hat{Q}_t, \hat{P}_t) \longrightarrow (\hat{q}_t, \hat{p}_t)$. Each unitary transformation can be represented by its Weyl symbol. The Weyl representation for the normal form transformation was derived in [17], while the representation for the time evolution is the quasi-flow given by equations (4)–(6).

Unlike classical canonical transformations, however, the Weyl representation of the composition of the transformations is not equal to the composition of Weyl representations. Therefore, in order to obtain the general-form quasi-flow, a composition formula for Weyl symbols is needed. That is, given that the Weyl symbol of $\hat{A}$ with respect to the canonical set $\hat{Q}, \hat{P}$ is $A(z)$, and that the symbols of $\hat{Q}$ and $\hat{P}$ with respect to the set $\hat{q}, \hat{p}$ are $Z^\mu(z)$, to find the Weyl symbol $a(z)$ of $\hat{A}$ with respect to $\hat{q}, \hat{p}$. In the following we derive a semiclassical expansion of $a(z)$ to $O(\hbar^4)$, which, together with the previous results, yields the quasi-flow $z_t^\mu(z)$ of nonlinear oscillators in a general form.

The composition formula is derived directly from the definition of the Weyl symbol. By definition $a(z) = \mathrm{Tr}\,\hat{A}\Delta(z)$, where $\Delta(z) = \int \frac{\mathrm{d}z'}{h}\,\mathrm{e}^{\frac{i}{\hbar}(\hat{z}^\mu - z^\mu)z'_\mu}$ are the basis operators for the Weyl representation [19] with respect to the set $\hat{z}$, while $\hat{A} = \int \frac{\mathrm{d}z}{h} A(z)\tilde{\Delta}(z)$, where $\tilde{\Delta}(z)$ are the basis operators with respect to $\hat{Z}$. Therefore,

$$a(z) = \int \frac{\mathrm{d}z'}{h} A(z')\,\mathrm{Tr}\,\tilde{\Delta}(z')\Delta(z) = \int \mathrm{d}z' A(z')\delta(z';z), \tag{A.1}$$

where $\delta(z';z)$, the symbol of $\tilde{\Delta}(z')$ with respect to $\hat{z}$ divided by h, has the semiclassical expansion

$$\delta(z';z) = \left(1 - \hbar^2\left(\left(\frac{1}{16}\partial_k\partial_l Z^\mu \partial^k \partial^l Z^\nu\right)\partial_\mu\partial_\nu - \left(\frac{1}{24}\partial_k Z^\lambda \partial^k\partial^l Z^\mu \partial_l Z^\nu\right)\partial_\lambda\partial_\mu\partial_\nu\right)\right) \times \delta(z' - Z(z)) + O(\hbar^4). \tag{A.2}$$

Using this result back in equation (A.1) gives the composition rule $a(z) = \mathring{A}(z, Z(z))$,

$$\mathring{A}(z,z') = \left(1 - \hbar^2\left(\left(\frac{1}{16}\partial_k\partial_l Z^\mu \partial^k \partial^l Z^\nu\right)\partial_\mu\partial_\nu + \left(\frac{1}{24}\partial_k Z^\lambda \partial^k\partial^l Z^\mu \partial_l Z^\nu\right)\partial_\lambda\partial_\mu\partial_\nu\right)\right) A(z'). \tag{A.3}$$

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